

THREE-DIMENSIONAL TERMINAL TORIC FLIPS

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ABSTRACT. We describe three-dimensional terminal toric flips. We obtain the complete local description of three-dimensional terminal toric flips.

1. INTRODUCTION

The main purpose of this paper is to describe three-dimensional terminal toric flips.

Theorem 1.1 (cf. Theorems 3.1 and 4.1). *Let $\varphi : X \rightarrow Y$ be a small proper toric morphism such that X is a three-dimensional toric variety with only terminal singularities. Note that X is not assumed to be \mathbb{Q} -factorial. Let $C \simeq \mathbb{P}^1$ be an exceptional curve of φ . Assume that $-K_X \cdot C > 0$. Then one of the torus invariant points of X on C is non-singular and another one is a terminal quotient singularity. In particular, X is \mathbb{Q} -factorial and is not Gorenstein around C .*

By this result, we obtain the complete local description of three-dimensional terminal toric flips (see Theorem 3.1). The first author apologizes for the mistake in [F2, Example 4.4.2], where he claims that there exist three-dimensional non- \mathbb{Q} -factorial terminal toric flips. However, Theorem 1.1 implies that there are no such flips. This paper is based on the first author's private notes and the third author's master's thesis [T].

We summarize the contents of this paper. In Section 2, we describe three-dimensional terminal toric singularities. The results are well known to the experts. Section 3 gives the complete classification of three-dimensional \mathbb{Q} -factorial terminal toric flips. It is a supplement to [KMM] and [M, Example-Claim 14-2-5]. In Section 4, we prove that there are no three-dimensional non- \mathbb{Q} -factorial terminal toric flips. Theorem 4.1 is the main theorem of this paper. The proof depends on the results in Sections 2 and 3.

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Notation. We will work over \mathbb{C} , the complex number field, throughout this paper. Let $v_i \in N \simeq \mathbb{Z}^3$ for $1 \leq i \leq k$. Then the symbol $\langle v_1, v_2, \dots, v_k \rangle$ denotes the cone $\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \dots + \mathbb{R}_{\geq 0}v_k$ in $N_{\mathbb{R}} \simeq N \otimes_{\mathbb{Z}} \mathbb{R}$.

2. THREE-DIMENSIONAL TERMINAL TORIC SINGULARITIES

In this section, we characterize non- \mathbb{Q} -factorial affine toric threefolds with terminal singularities. We will use the same notation as in [YPG], which is an excellent exposition on terminal singularities.

Let X be an affine toric threefold. First, let us recall the following well-known theorem of G. K. White, D. Morrison, G. Stevens, V. Danilov, and M. Frumkin (see [YPG, (5.2) Theorem]).

Theorem 2.1. *Assume that X is \mathbb{Q} -factorial. Then X is terminal if and only if (up to permutations of (x, y, z) and symmetries of μ_r) $X \simeq \mathbb{C}^3/\mu_r$ of type $\frac{1}{r}(a, -a, 1)$ with a coprime to r , where μ_r is the cyclic group of order r . In particular, if X is Gorenstein and terminal, then X is non-singular.*

Here, we prove the following well-known result for the reader's convenience (cf. [I], and [II, Theorem 3.6]).

Theorem 2.2. *Assume that X is not \mathbb{Q} -factorial. Then X is terminal if and only if $X \simeq \text{Spec } \mathbb{C}[x, y, z, w]/(xy - zw)$. We call this singularity an ordinary double point.*

By the above theorems, we obtain the complete list of three-dimensional terminal toric singularities.

Remark 2.3. Mori classified three-dimensional terminal singularities. For the details, see [YPG, (6.1) Theorem]. We do not use his classification table in this paper.

Proof of Theorem 2.2. Let $N = \mathbb{Z}^3$ and $\Delta = \langle e_1, \dots, e_k \rangle$ the cone in N such that $X = X(\Delta)$, where each e_i is primitive. First, we prove

Claim 1. *If X is non- \mathbb{Q} -factorial and terminal, then $k = 4$.*

Proof of the claim. It is obvious that $k \geq 4$. Since X is \mathbb{Q} -Gorenstein, there is a hyperplane $H \subset N$ that contains every e_i . On $H \simeq \mathbb{Z}^2$, e_i s span two dimensional convex polygon P . By renumbering e_i s, we can assume that they are arranged counter-clockwise. Since $X(\Delta)$ is terminal, all the lattice points in P are e_i s. In particular, the triangle on H spanned by e_1 , e_2 , and e_3 contains only three lattice points e_i ($1 \leq i \leq 3$) of H . So, after changing the coordinate of H , we can assume that $e_1 = (0, 1)$, $e_2 = (0, 0)$, and $e_3 = (1, 0)$ in $H \simeq \mathbb{Z}^2$. It can be checked easily that $(1, 1) \in P$ since $k \geq 4$. Thus, we obtain that $k = 4$ and $e_4 = (1, 1)$. \square

Claim 2. *Assume that X is non- \mathbb{Q} -factorial, Gorenstein, and terminal. Then X is isomorphic to $\text{Spec } \mathbb{C}[x, y, z, w]/(xy - zw)$.*

Proof of the claim. On this assumption, the cones $\langle e_1, e_2, e_3 \rangle$, $\langle e_1, e_2, e_4 \rangle$, $\langle e_1, e_3, e_4 \rangle$, and $\langle e_2, e_3, e_4 \rangle$ define \mathbb{Q} -factorial Gorenstein affine toric threefolds with terminal singularities. By Theorem 2.1, every cone listed above is non-singular. So, by changing the coordinate of N , we can assume that $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Since X is Gorenstein and $\langle e_1, e_2, e_4 \rangle$, $\langle e_1, e_3, e_4 \rangle$, and $\langle e_2, e_3, e_4 \rangle$ are non-singular, $e_4 = (-1, 1, 1)$, $(1, -1, 1)$, or $(1, 1, -1)$. Anyway, we can check that $X \simeq \text{Spec } \mathbb{C}[x, y, z, w]/(xy - zw)$. \square

By the above claim, it is sufficient to prove

Claim 3. *All the non- \mathbb{Q} -factorial toric affine threefolds with terminal singularities are Gorenstein.*

Proof of the claim. We assume that X is not Gorenstein and obtain a contradiction.

Let \overline{N} be the sublattice of N spanned by all the lattice points on H and the origin of N . In \overline{N} , $\Delta = \langle e_1, e_2, e_3, e_4 \rangle$ defines a Gorenstein terminal threefold. So, we can assume that $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and $e_4 = (1, 1, -1) \in \mathbb{Z}^3 \simeq \overline{N}$ by the proof of Claim 2. First, we consider $\langle e_1, e_2, e_3 \rangle$ in \overline{N} and N . By Theorem 2.1, we obtain that $N = \overline{N} + \mathbb{Z} \cdot \frac{1}{r}(\alpha, \beta, \gamma)$, where (α, β, γ) is one of the followings: $(a, -a, 1)$, $(a, 1, -a)$, $(-a, a, 1)$, $(-a, 1, a)$, $(1, a, -a)$, $(1, -a, a)$ such that $0 < a < r$ with a coprime to r . Next, we use the terminality of $\langle e_1, e_2, e_4 \rangle$. We consider the linear transform $T : N \rightarrow N$ such that $Te_1 = e_1$, $Te_2 = e_2$, $Te_4 = e_3$. Then $TN = T\overline{N} + \mathbb{Z} \cdot \frac{1}{r}(\alpha', \beta', \gamma')$, where $(\alpha', \beta', \gamma')$ is one of the followings: $(1 + a, 1 - a, -1)$, $(0, 1 - a, a)$,

$(1 - a, 1 + a, -1)$, $(0, 1 + a, -a)$, $(1 - a, 0, a)$, $(1 + a, 0, -a)$. Note that

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

We treat the first case, that is, $(\alpha', \beta', \gamma') = (1 + a, 1 - a, -1)$. By the terminal lemma (see [YPG, (5.4) Theorem]), r divides $(1 + a) + (1 - a) = 2$ since it does not divide $(1 + a) + (-1)$ nor $(1 - a) + (-1)$. So, $r = 2$ and $a = 1$. Thus $\frac{1}{r}(\alpha', \beta', \gamma') = \frac{1}{2}(2, 0, -1) \equiv \frac{1}{2}(0, 0, 1) \pmod{T\bar{N}}$. It is a contradiction (see Theorem 2.1). We leave the other cases for the reader's exercise. So, there are no non-Gorenstein non- \mathbb{Q} -factorial affine toric threefolds with terminal singularities. \square

Therefore, we completed the proof of Theorem 2.2. \square

Theorem 2.2 has a beautiful corollary.

Corollary 2.4 (Three-dimensional terminal toric flop). *Let*

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & W & \end{array}$$

be a three-dimensional toric flopping diagram such that W is affine. Assume that X has only terminal singularities. Then it is the simplest flop, where the simplest flop means the flop described in [Fl, p.49–p.50].

Proof. By the assumption, W is a non- \mathbb{Q} -factorial affine toric threefold with terminal singularities. Thus, $X \simeq \operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - zw)$ by Theorem 2.2. So, the above diagram must be the simplest flop. \square

3. THREE-DIMENSIONAL \mathbb{Q} -FACTORIAL TERMINAL TORIC FLIPS

We classify three-dimensional flipping contractions from \mathbb{Q} -factorial terminal toric threefolds. The next theorem was stated in [KMM] without proof at the end of Example 5-2-5.

Theorem 3.1 (Three-dimensional \mathbb{Q} -factorial terminal toric flips). *Let $\varphi_R : X(\Delta) \rightarrow Y(\Sigma)$ be the contraction morphism of an extremal ray R with $K_X \cdot R < 0$ of flipping type from a toric threefold with only \mathbb{Q} -factorial terminal singularities. Assume that Y is affine. Then we have the following description of the flipping contraction:*

There exist two three-dimensional cones

$$\begin{aligned} \tau_4 &= \langle v_1, v_2, v_3 \rangle \in \Delta, \\ \tau_3 &= \langle v_1, v_2, v_4 \rangle \in \Delta, \end{aligned}$$

sharing the two-dimensional wall

$$w = \langle v_1, v_2 \rangle$$

such that $[V(w)] \in R$ and that for some \mathbb{Z} -coordinate of $N \simeq \mathbb{Z}^3$,

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (0, 0, 1), \\ v_4 &= (a, r - a, -r), \end{aligned}$$

or

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (0, 0, 1), \\ v_4 &= (a, 1, -r), \end{aligned}$$

where $0 < a < r$ and $\gcd(r, a) = 1$. Therefore,

$$\Delta = \{\tau_3, \tau_4, \text{and their faces}\},$$

and

$$\Sigma = \{\langle v_1, v_2, v_3, v_4 \rangle, \text{and its faces}\}.$$

Proof. By [M, Example-Claim 14-2-5], it is sufficient to prove that the (unique) rational curve that is contracted passes through only one singular point of X . Without loss of generality, we may assume that $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ since $\langle v_1, v_2 \rangle$ is a two-dimensional non-singular cone. Seeking a contradiction, we assume that both $\langle v_1, v_2, v_3 \rangle$ and $\langle v_1, v_2, v_4 \rangle$ are singular. By the terminal lemma ([O, §1.6]), we may assume that $v_3 = (1, p, q)$, where $0 < p < q$ and $\gcd(p, q) = 1$. We note that $q \geq 2$. We can write $v_4 = av_1 + bv_2 + c(k, l, -1)$ with $0 < a < c$, $0 < b < c$, $\gcd(a, c) = 1$, $\gcd(b, c) = 1$, and $k, l \in \mathbb{Z}$. In particular, $c \geq 2$. We note that we assumed that $\langle v_1, v_2, v_4 \rangle$ is singular and terminal. By the terminal lemma again (see [O, p.36 White's Theorem]), at least one of $a - 1$, $b - 1$ and $a + b$ is divisible by c . Therefore, $a = 1$, $b = 1$, or $a + b = c$. We note that v_1, v_2, v_3 are on the plane

$$x + y - \frac{p}{q}z = 1.$$

Case 1 ($a = 1$). In this case, $v_4 = (1 + ck, b + cl, -c)$. We have

$$\frac{c}{q}v_3 + v_4 = (1 + ck + \frac{c}{q}, b + cl + \frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$(1) \quad 1 + ck + \frac{c}{q} > 0,$$

$$(2) \quad b + cl + \frac{p}{q}c > 0,$$

and

$$(3) \quad 1 + ck + b + cl + \frac{p}{q}c < 1.$$

The inequalities (1) and (2) follow from the condition that φ_R is small. The condition $K_X \cdot R < 0$ implies the inequality (3). By (2) and (3), we have $k \leq -1$. Thus

$$0 < 1 + ck + \frac{c}{q} \leq 1 - c + \frac{c}{q} \leq 1 - \frac{1}{2}c \leq 0$$

by (1). It is a contradiction.

Case 2 ($b = 1$). In this case, $v_4 = (a + ck, 1 + cl, -c)$. We have

$$\frac{c}{q}v_3 + v_4 = (a + ck + \frac{c}{q}, 1 + cl + \frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$(4) \quad a + ck + \frac{c}{q} > 0,$$

$$(5) \quad 1 + cl + \frac{p}{q}c > 0,$$

and

$$(6) \quad a + ck + 1 + cl + \frac{p}{q}c < 1.$$

By (5) and (6), $k \leq -1$. So, $k = -1$ by (4). By (5), we know that $l \geq -1$. Therefore, $l = 0$ or -1 by (6).

First, we assume that $l = 0$. Then we get

$$a - c + \frac{p}{q}c < 0$$

by (6) and

$$a - c + \frac{c}{q} > 0$$

by (4). It is a contradiction.

Next, we assume that $l = -1$. Then we obtain

$$a - c + \frac{c}{q} > 0$$

by (4) and

$$1 - c + \frac{p}{q}c > 0$$

by (5). These two inequalities imply that

$$1 + a - 2c + \frac{p+1}{q}c > 0.$$

It is a contradiction.

Case 3 ($a + b = c$). In this case, $v_4 = (a + ck, c - a + cl, -c)$. We have

$$\frac{c}{q}v_3 + v_4 = (a + ck + \frac{c}{q}, c - a + cl + \frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$(7) \quad a + ck + \frac{c}{q} > 0,$$

$$(8) \quad c - a + cl + \frac{p}{q}c > 0,$$

and

$$(9) \quad a + ck + c - a + cl + \frac{p}{q}c < 1.$$

By (8) and (9), $k \leq -1$. So, $k = -1$ by (7). By (8), we have $l \geq -1$. Therefore, $l = 0$ or -1 by (9).

First, we assume that $l = 0$. Then we have

$$\frac{p}{q}c < 1$$

by (9) and

$$a - c + \frac{c}{q} > 0$$

by (7). Thus,

$$1 > \frac{p}{q}c \geq \frac{c}{q} > c - a \geq 1.$$

It is a contradiction.

Next, we assume that $l = -1$. Then we obtain

$$a - c + \frac{c}{q} > 0$$

by (7) and

$$-a + \frac{p}{q}c > 0$$

by (8). By adding these two inequalities, we have

$$-c + \frac{p+1}{q}c > 0.$$

It is a contradiction.

Therefore, at least one of $\langle v_1, v_2, v_3 \rangle$ and $\langle v_1, v_2, v_4 \rangle$ must be non-singular. Thus, we have the desired description of $\varphi_R : X \rightarrow Y$ by [M, Example-Claim 14-2-5]. \square

Remark 3.2. The example in [M, Remark 14-2-7 (ii)] is not true. The cone $\langle v_1, v_2, v_3 \rangle$ is not terminal. The cone $\langle v_1, v_2, v_3 \rangle$ has canonical singularities.

Remark 3.3. The source space X in Theorem 3.1 is always singular.

Remark 3.4. In [M, Example-Claim 14-2-5], X is assumed to be *complete*. It is because contraction morphisms of extremal rays are constructed only for *complete* varieties in [R] and [M, Chapter 14]. For the details of non-complete toric varieties, see [FS], [F1], and [S].

4. MAIN THEOREM

The following theorem is the main theorem of this paper.

Theorem 4.1 (cf. [T]). *Let $\varphi : X \rightarrow Y$ be a small proper toric morphism such that X is a three-dimensional toric variety with only terminal singularities. Let $C \simeq \mathbb{P}^1$ be an exceptional curve of φ . Assume that $-K_X \cdot C > 0$. Then C does not pass through ordinary double points.*

Proof. First, we assume that C passes through two ordinary double points. By taking a small projective resolution of X , we can assume that C does not pass through any singular points. It is a contradiction by Theorem 3.1 (see Remark 3.3).

Next, we assume that C passes through only one ordinary double points. By Theorem 3.1, we have the following local description of X and C :

There exist lattice points of $N = \mathbb{Z}^3$

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (0, 1, 0), & v_3 &= (0, 0, 1), \\ v_5 &= (-1, 1, 1), & v_6 &= (1, -1, 1). \end{aligned}$$

We put

$$\Delta_1 = \{\langle v_1, v_2, v_3, v_5 \rangle, \langle v_1, v_2, v_4 \rangle, \text{and their faces}\},$$

and

$$\Delta_2 = \{\langle v_1, v_2, v_3, v_6 \rangle, \langle v_1, v_2, v_4 \rangle, \text{and their faces}\},$$

where $v_4 = (a, r-a, -r)$ or $(a, 1, -r)$ with $0 < a < r$ and $\gcd(a, r) = 1$. Then $X = X(\Delta)$, where $\Delta = \Delta_1$ or Δ_2 , and C is $V(\langle v_1, v_2 \rangle) \simeq \mathbb{P}^1$.

Case 1. When $v_4 = (a, r-a, -r)$ and $\Delta = \Delta_1$, we have

$$v_2 = \frac{r}{2r-a}v_5 + \frac{r-a}{2r-a}v_1 + \frac{1}{2r-a}v_4.$$

Therefore, v_2 is contained in the cone $\langle v_5, v_1, v_4 \rangle$. Thus, we can not remove the wall $\langle v_1, v_2 \rangle$ from Δ .

Case 2. When $v_4 = (a, 1, -r)$ and $\Delta = \Delta_1$, we have

$$v_2 = \frac{r}{r+1}v_5 + \frac{r-a}{r+1}v_1 + \frac{1}{r+1}v_4.$$

Therefore, v_2 is contained in the cone $\langle v_5, v_1, v_4 \rangle$. Thus, we can not remove the wall $\langle v_1, v_2 \rangle$ from Δ .

Case 3. When $v_4 = (a, r-a, -r)$ and $\Delta = \Delta_2$, we have

$$v_1 = \frac{r}{r+a}v_6 + \frac{a}{r+a}v_2 + \frac{1}{r+a}v_4.$$

Therefore, v_1 is contained in the cone $\langle v_6, v_2, v_4 \rangle$. Thus, we can not remove the wall $\langle v_1, v_2 \rangle$ from Δ .

Case 4. When $v_4 = (a, 1, -r)$ and $\Delta = \Delta_2$, we have

$$v_1 = \frac{r}{r+a}v_6 + \frac{r-1}{r+a}v_2 + \frac{1}{r+a}v_4.$$

Therefore, v_1 is contained in the cone $\langle v_6, v_2, v_4 \rangle$. Thus, we can not remove the wall $\langle v_1, v_2 \rangle$ from Δ .

Thus, C does not pass through any ordinary double points. \square

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